Lecture 1 — Model Categories: Definitions

Relative categories and localization

Definition. A subcategory $W \subseteq C$ is **wide** if $\mathrm{ob}W = \mathrm{ob}C$, i.e. W contains every object of C.

Definition. A **relative category** is a pair (C, W) with C a category and W a wide subcategory. A **relative functor** $F\colon (C, W) \to (D, V)$ is a functor $F\colon C \to D$ such that $F(W) \subseteq V$.

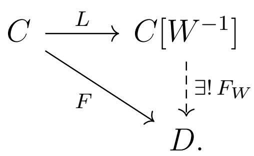
Definition. A relative category (C, W) is a **category with weak equivalences** if:

1. W contains all isomorphisms of C.
2. **(2-out-of-3)** For composable morphisms f, g in C, if any two of f, g, gf belong to W, so does the third.

Morphisms in W are called **weak equivalences**.

Definition 1. Let (C, W) be a category with weak equivalences. A **localization** of C at W is a functor $L\colon C \to C[W^{-1}]$ such that $L(W) \subseteq \mathrm{Core}(C[W^{-1}])$, satisfying:

1. **(Universal property.)** For every functor $F\colon C \to D$ with $F(W) \subseteq \mathrm{Core}(D)$, there is a unique $F_W\colon C[W^{-1}] \to D$ with $F_W \circ L = F$:



2. **(Full faithfulness.)** For every category D, precomposition with L gives a fully faithful functor
$$L^*\colon C[W^{-1}]^D \longrightarrow C^D.$$

Definition. The **explicit construction** of $C[W^{-1}]$ is as follows. Let $F(C, W^{-1})$ be the category with $\mathrm{ob}F(C, W^{-1}) = \mathrm{ob}C$, and morphisms from X to Y given by finite composable strings (zig-zags) $f_1, f_2, \ldots, f_n$ where each f_i is either a morphism of C (a forward step $\to$) or a formal inverse w_i^{-1} of some $w_i \in W$ (a backward step $\leftarrow$). The empty string is Id_X. Set $C[W^{-1}] = F(C, W^{-1})/\!\sim$ where $\sim$ is generated by:

$$[\mathrm{Id}_A] = \mathrm{Id}_A, \qquad [g, f] = [g \circ f], \qquad [w, w^{-1}] = \mathrm{Id}_{s(w)}, \qquad [w^{-1}, w] = \mathrm{Id}_{t(w)},$$

with $f, g \in \mathrm{Mor}C$ composable, $w \in W$.

Remark (Size problem). *In general $C[W^{-1}]$ is* not *locally small: the class of zig-zag paths between two fixed objects X and Y is not bounded, since zig-zags can pass through class-many intermediate objects and grow to arbitrary length.*

Model categories: definitions

Definition. The **arrow category** $\operatorname{Map} C$ of C has:

- Objects: morphisms $A \xrightarrow{f} B$ of C.
- Morphisms from $f\colon A \to B$ to $g\colon C \to D$: commutative squares

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D. \end{array}$$

Definition. A morphism $f\colon A \to B$ is a **retract** of $g\colon C \to D$ if f is a retract of g as objects of $\operatorname{Map} C$:

$$\begin{array}{ccccc} & & \overset{\mathrm{Id}_A}{\frown} & & \\ A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \\ & & \underset{\mathrm{Id}_B}{\smile} & & \end{array}$$

Definition. A **functorial factorization** on C is a functor $(\alpha, \beta)\colon \operatorname{Map} C \to \operatorname{Map} C \times \operatorname{Map} C$, $f \mapsto (\alpha f, \beta f)$, with $s(\alpha f) = s(f)$, $t(\beta f) = t(f)$, $t(\alpha f) = s(\beta f)$, and $f = \beta f \circ \alpha f$:

$$\cdot \xrightarrow{\alpha f} \cdot \xrightarrow{\beta f} \cdot, \qquad f = \beta f \circ \alpha f.$$

Definition. Let $i\colon A \to B$ and $p\colon X \to Y$. i has the **left lifting property** (LLP) with respect to p (equivalently, p has the **right lifting property** with respect to i), written $i \boxslash p$, if for every commutative square there exists a diagonal filler:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & {\scriptstyle \exists h}\nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array} \qquad h \circ i = f, \; p \circ h = g.$$

For a class S: $\mathrm{LLP}(S) = \{i \mid i \boxslash p \ \forall p \in S\}$ and $\mathrm{RLP}(S) = \{p \mid i \boxslash p \ \forall i \in S\}$.

Definition 2. A **model structure** on C consists of wide subcategories W, Cof, Fib and two functorial factorizations (α, β), (γ, δ), satisfying:

1. **(2-out-of-3)** (C, W) is a category with weak equivalences.
2. **(Retracts)** W, Cof, Fib are each closed under retracts.
3. **(Lifting)** $\mathrm{Cof} \subseteq \mathrm{LLP}(W \cap \mathrm{Fib})$ and $\mathrm{Fib} \subseteq \mathrm{RLP}(W \cap \mathrm{Cof})$.
4. **(Factorization)** For every f:

$$\alpha f \in \mathrm{Cof}, \quad \beta f \in W \cap \mathrm{Fib} \qquad \text{and} \qquad \gamma f \in W \cap \mathrm{Cof}, \quad \delta f \in \mathrm{Fib}.$$

Morphisms in W / Cof / Fib are **weak equivalences** / **cofibrations** / **fibrations**. Those in $W \cap \mathrm{Cof}$ (resp. $W \cap \mathrm{Fib}$) are **trivial (acyclic) cofibrations** (resp. **trivial (acyclic) fibrations**).

Definition 3. A **model category** is a bicomplete category C equipped with a model structure.

Example.

1. $W = \mathrm{Iso}(C)$, $\mathrm{Cof} = \mathrm{Fib} = \mathrm{Mor}C$ is a model structure on any bicomplete C.

2. If C and D are model categories, so is $C \times D$ with the componentwise model structure.

3. For any model category C, the opposite C^{op} is a model category with fibrations and cofibrations exchanged. **Every statement has a dual statement.**

Definition. In any category C: $0 = \operatorname{colim} \varnothing$ (initial object) and $* = \lim \varnothing$ (terminal object). For every X there are unique morphisms

$$0 \xrightarrow{i} X \xrightarrow{t} *.$$

Definition 4. Let C be a model category.

- X is **cofibrant** if $i\colon 0 \to X \in \mathrm{Cof}$.
- X is **fibrant** if $t\colon X \to * \in \mathrm{Fib}$.

Write C_c, C_f, C_{cf} for the full subcategories of cofibrant, fibrant, and cofibrant-and-fibrant objects.

Definition 5. Let C be a model category.

- The **cofibrant replacement** of X is QX, obtained by factoring $0 \to X$ via (α, β):

$$0 \xrightarrow{\in \mathrm{Cof}} QX \xrightarrow[q_X]{\in W \cap \mathrm{Fib}} X.$$

- The **fibrant replacement** of X is RX, obtained by factoring $X \to *$ via (γ, δ):

$$X \xrightarrow[r_X]{\in W \cap \mathrm{Cof}} RX \xrightarrow{\in \mathrm{Fib}} *.$$

Both Q and R are functors. The diagrams are called the **(co)fibrant replacement** of X.

Lemma 6 (Retract Argument)**.** *Suppose $f = p \circ i$ in C.*

1. *If $f \boxslash p$, then f is a retract of i.*
2. *Dually, if $i \boxslash f$ then f is a retract of p.*

Proof. Write $f\colon A \to C$, $i\colon A \to B$, $p\colon B \to C$. The lifting hypothesis applied to the square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & \overset{\exists r}{\nearrow} & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\mathrm{Id}_C]{} & C \end{array}$$

yields $r\colon C \to B$ satisfying $r \circ f = i$ (upper triangle) and $p \circ r = \mathrm{Id}_C$ (lower triangle). The retract diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{Id}_A} & A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[r]{} & B & \xrightarrow[p]{} & C \end{array}$$

shows f is a retract of i: horizontal composites are Id_A (top) and $p \circ r = \mathrm{Id}_C$ (bottom); the left square commutes by $r \circ f = i$; the right square commutes since $f = p \circ i$. Part (2) is dual. □

Lemma 7. *In a model category:*

$$\begin{aligned} f \in \mathrm{Cof} &\iff f \in \mathrm{LLP}(W \cap \mathrm{Fib}), \\ f \in W \cap \mathrm{Cof} &\iff f \in \mathrm{LLP}(\mathrm{Fib}), \\ f \in \mathrm{Fib} &\iff f \in \mathrm{RLP}(W \cap \mathrm{Cof}), \\ f \in W \cap \mathrm{Fib} &\iff f \in \mathrm{RLP}(\mathrm{Cof}). \end{aligned}$$

Proof. We prove $\mathrm{Cof} = \mathrm{LLP}(W \cap \mathrm{Fib})$; the rest are analogous or dual.

($\subseteq$): The inclusion $\mathrm{Cof} \subseteq \mathrm{LLP}(W \cap \mathrm{Fib})$ is the first clause of Axiom 3 of Definition 2.

($\supseteq$): Let $f \in \mathrm{LLP}(W \cap \mathrm{Fib})$. Factor $f = p \circ i$ with $i \in \mathrm{Cof}$, $p \in W \cap \mathrm{Fib}$. Since $f \boxslash p$, the Retract Argument (Lemma 6) gives f as a retract of i. As Cof is closed under retracts, $f \in \mathrm{Cof}$. □

Remark. *Any two of* W, Cof, Fib *determine the third:*

- W *and* Cof*:* $\mathrm{Fib} = \mathrm{RLP}(W \cap \mathrm{Cof})$.
- W *and* Fib*:* $\mathrm{Cof} = \mathrm{LLP}(W \cap \mathrm{Fib})$.
- Cof *and* Fib*: a map f lies in W if and only if it factors as $f = p \circ i$ with* $i \in \mathrm{LLP}(\mathrm{Fib}) = W \cap \mathrm{Cof}$ *and* $p \in \mathrm{RLP}(\mathrm{Cof}) = W \cap \mathrm{Fib}$.

Corollary 8. *In a model category:*

- Cof *and* $W \cap \mathrm{Cof}$ *are closed under pushout.*
- Fib *and* $W \cap \mathrm{Fib}$ *are closed under pullback.*

Lemma 9 (Ken Brown)**.** *Let C be a model category, (D, W_D) a category with weak equivalences, and $F\colon C \to D$ a functor. Suppose $F(f) \in W_D$ for every $f \in W \cap \mathrm{Cof}$ whose domain and codomain are both cofibrant. Then F sends every weak equivalence between cofibrant objects to W_D.*

Proof. Let $f\colon A \to B$ with $f \in W$ and A, B cofibrant.

We factor $(f, \mathrm{Id}_B)\colon A \sqcup B \to B$. Apply (α, β) to (f, Id_B):

$$\begin{array}{ccc} A \sqcup B & \xrightarrow{(f,\mathrm{Id}_B)} & B \\ & {\scriptstyle q\,\in\mathrm{Cof}}\searrow \quad \nearrow{\scriptstyle p\,\in W\cap\mathrm{Fib}} & \\ & C. & \end{array}$$

A and B cofibrant means $0 \to A$ and $0 \to B$ are in Cof. The pushout square (with Cof closed under pushout by Corollary 8):

$$\begin{array}{ccc} 0 & \xrightarrow{\in\mathrm{Cof}} & A \\ {\scriptstyle\in\mathrm{Cof}}\downarrow & & \downarrow{\scriptstyle i_1\,\in\mathrm{Cof}} \\ B & \xrightarrow[i_2\,\in\mathrm{Cof}]{} & A \sqcup B. \end{array}$$

Hence $q \circ i_1, q \circ i_2 \in \mathrm{Cof}$. Moreover C is cofibrant, since $0 \to A \to A \sqcup B \xrightarrow{q} C$ is a composite of cofibrations.

- $p \circ (q \circ i_1) = f \in W$ and $p \in W \cap \mathrm{Fib} \subseteq W$; by 2-of-3, $q \circ i_1 \in W$. So $q \circ i_1 \in W \cap \mathrm{Cof}$.
- $p \circ (q \circ i_2) = \mathrm{Id}_B \in W$ and $p \in W$; by 2-of-3, $q \circ i_2 \in W$. So $q \circ i_2 \in W \cap \mathrm{Cof}$.

Both have cofibrant domain (A resp. B) and cofibrant codomain C, so by hypothesis $F(q \circ i_1), F(q \circ i_2) \in W_D$.

From $F(p) \circ F(q \circ i_2) = F(\mathrm{Id}_B) \in W_D$ and $F(q \circ i_2) \in W_D$, the 2-of-3 in D gives $F(p) \in W_D$. Finally,

$$F(f) \;=\; F(p) \circ F(q \circ i_1) \;\in\; W_D.$$

□